

CONTINUITY OF TOPOLOGICAL ENTROPY FOR PERTURBATION OF TIME-ONE MAPS OF HYPERBOLIC FLOWS

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ABSTRACT. We consider a C^1 neighborhood of the time-one map of a hyperbolic flow and prove that the topological entropy varies continuously for diffeomorphisms in this neighborhood. This shows that the topological entropy varies continuously for all known examples of partially hyperbolic diffeomorphisms with one-dimensional center bundle.

1. INTRODUCTION

In this paper we are addressing the problem of continuity of the topological entropy on the set of partially hyperbolic diffeomorphisms with the dimension of the center equal to one, and in particular for perturbations of time-one maps of hyperbolic flows. The main theorem which we obtain is the following.

Theorem A. *Let ϕ_t be a C^1 Anosov flow on a compact Riemannian manifold M . There exists a C^1 neighborhood \mathcal{U} of ϕ_1 , in the set of C^1 diffeomorphisms of M , such that the topological entropy varies continuously in \mathcal{U} .*

The topological entropy, together with the metric entropy, is an important invariant of a dynamical system (see the definition in the following section), which describes the complexity of the orbits. The dependence of the topological entropy on the map was extensively studied by various mathematicians (Newhouse, Yomdin, Katok, etc.). There are several important factors which contribute to this dependence, namely the manifold, and the space of maps which is considered.

Let us remark that in the space of continuous maps, generically the topological entropy is infinity ([28]), so it make sense to study the problem of continuity at least in the C^1 topology.

The upper semicontinuity of the topological entropy seems to depend on the regularity of the maps and on the existence of homoclinic tangencies. By Yomdin [29] (see also [5]), the topological entropy is upper semicontinuous in the space of C^∞ diffeomorphisms. And by Misiurewicz [20], the topological entropy is not upper semicontinuous in the C^r topology, for any $r < \infty$. The examples where

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the upper semicontinuity fails are diffeomorphisms with homoclinic tangencies; an arbitrary C^r small perturbation can create a horseshoe at the homoclinic orbit, such that the topological entropy increases (proportionally with $\frac{1}{r}$). We note that, by the result of Liao, Viana and Yang [17], the topological entropy is in fact upper semicontinuous on the space of C^1 diffeomorphisms away from tangencies.

The lower continuity of the topological entropy seems to depend on the existence of horseshoes or other types of invariant sets which are robust and “carry” the entropy (for example uniformly hyperbolic sets), and in many cases this depends on the dimension of the manifold M . In dimension one, the topological entropy is lower semicontinuous on the space of C^0 maps and locally constant (zero) on the space of homeomorphisms, while in dimension two it is lower semicontinuous on the space of $C^{1+\alpha}$ diffeomorphisms, by a result of Katok ([15]). The reason for these results is that in lower dimensions the topological entropy is due to the existence of horseshoes, and these horseshoes are persistent after small perturbations. In dimension greater or equal to three one can easily construct examples of C^∞ diffeomorphisms where the lower semicontinuity of the topological entropy fails, using partially hyperbolic invariant sets with positive entropy that can disappear after an arbitrarily small perturbation.

Of course, the classical example of diffeomorphisms which are both away from tangencies and have persistent horseshoes are the uniformly hyperbolic maps, in this case the topological entropy is not only continuous, it is locally constant in the C^1 topology due to the structural stability. The partially hyperbolic diffeomorphisms, as a generalization of uniformly hyperbolic maps, were proposed by Brin, Pesin [4] and by Pugh, Shub [24] independently in the early 1970’s. A diffeomorphism f is *partially hyperbolic*, if there exists a decomposition $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle TM into three continuous sub-bundles $x \mapsto E_x^s$, $x \mapsto E_x^c$ and $x \mapsto E_x^u$ such that

- all three sub-bundles E^s , E^c and E^u are invariant under the derivative Df ;
- $Df|_{E^s}$ is a uniform contraction, $Df|_{E^u}$ is a uniform expansion and $Df|_{E^c}$ lies in between them: there exists $\mu < 1$ such that

$$\frac{\|Df(x)v^s\|}{\|Df(x)v^c\|} \leq \mu \quad \text{and} \quad \frac{\|Df(x)v^c\|}{\|Df(x)v^u\|} \leq \mu$$

for any unit vectors $v^s \in E^s$, $v^c \in E^c$ and $v^u \in E^u$ and any $x \in M$.

Partially hyperbolic diffeomorphisms form an open subset of the space of C^r diffeomorphisms of M , for any $r \geq 1$.

A natural question to ask is how is the dependence of the topological entropy on the space of partially hyperbolic diffeomorphisms with the dimension of the center equal to one or two, since the topological entropy is constant zero for homeomorphisms in dimension one and continuous for C^∞ diffeomorphisms in dimension two. An example of a C^∞ partially hyperbolic diffeomorphisms with 2-D center where the topological entropy is not lower semicontinuous can be found in [14] (an invariant set with high entropy can disappear after an arbitrarily small perturbation). Thus, we will focus on the partially hyperbolic diffeomorphisms with one dimensional center, and a natural conjecture formulated in this setting is the following.

Conjecture 1. The topological entropy is continuous on the space of C^1 partially hyperbolic diffeomorphisms with the dimension of the center equal to one.

An easier version would be to restrict the attention to the space of three dimensional manifolds, where the stable, center, and unstable bundle of a partially hyperbolic diffeomorphism are automatically one dimensional.

Conjecture 2. The topological entropy depends continuously in the space of 3 dimensional partially hyperbolic diffeomorphisms.

We remark that these conjectures are known to be true for all the known examples of partially hyperbolic diffeomorphisms with one dimensional center or in dimension three. A list with the known partially hyperbolic diffeomorphisms with one dimensional center is the following:

- (1) uniformly hyperbolic diffeomorphisms
- (2) skew products over uniformly hyperbolic, with the fiber being a circle, and perturbations ([2, 21, 9, 10])
- (3) derived from Anosov diffeomorphisms ([18, 10, 23])
- (4) non-dynamically coherent examples of Hertz-Hertz-Ures [26],
- (5) time-one maps of Anosov flows and perturbations
- (6) new examples by Bonatti-Parwani-Potrie [1]

In the first 4 examples, the topological entropy is locally constant. In the cases (5) and (6), the topological entropy is not locally constant. In fact, in case (5), let X_t be a C^∞ Anosov flow, and considering the smooth family of time- t diffeomorphisms $f_t = X_t$, we observe that $h(f_t) = th(X)$ is not constant. The diffeomorphism in case (6) has a unique attracting set and a unique expanding set, and restricted to the attracting set (resp. expanding set) the dynamic is that of (a perturbation of) the time-one map of a hyperbolic flow restricted to an attractor (resp. repeller), hence the discussion is similar to the case (5) from above for time-one maps of Anosov flows. More details on case (6) will be given in Appendix. The continuity of the topological entropy at the time-one map of a transitive Anosov flow was proven by Hua in her thesis [13], and for some specific perturbations by Hu-Zhu [12]; however, they only show the continuity at one point (the time-one map of the flow), and not for a neighborhood. The result of our paper deals with the case of perturbations of the time-one maps of Anosov flows, and similar arguments can be used also in the case of Bonatti-Parwani-Potrie examples, which will be explained in Appendix, thus completing all the known cases mentioned above.

Our proof of the lower semicontinuity of topological entropy is different from the method used in most other papers, namely the construction of horseshoes, because in our case they may not exist. The main idea of our paper (and the most difficult part) is to show that the unstable foliation has uniform expansion on every unstable disk. Under these conditions of uniformity one can show that this expansion cannot drop much after small perturbations, and the connection with the topological entropy will then help us get the final conclusion. In this last argument the fact that the center bundle has dimension one is essential.

The paper is organized as follows. In Section 2 we introduce the definitions and various preliminary results, while in Section 3 we give the proof of Theorem A. In the Appendix we include a discussion on the other known examples of partially hyperbolic diffeomorphisms with the dimension of the center equal to one.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. Time-one maps of Anosov flows and perturbations. We start by introducing the notion of hyperbolic flow and the spectral decomposition.

Definition 2.1. A C^1 flow ϕ on the compact Riemannian manifold M is *Anosov* (or *uniformly hyperbolic*) if there exists a splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ which is invariant under the flow, E^c is the direction of the flow, and there exist $C, \lambda > 0$ such that

$$\|D\phi_t(x)v^s\| \leq C \exp^{-\lambda t} \|v^s\|, \quad \forall x \in M, t > 0, v^s \in E_x^s;$$

$$\|D\phi_t(x)v^u\| \geq C^{-1} \exp^{\lambda t} \|v^u\|, \quad \forall x \in M, t > 0, v^u \in E_x^u.$$

The map $\phi_t : M \rightarrow M$ is the *time- t map of the flow* ϕ .

We remark that the time- t map of an Anosov flow is a partially hyperbolic diffeomorphism, meaning that the tangent bundle has an invariant splitting $TM = E^s \oplus E^c \oplus E^u$ (in this case the same one of the Anosov flow), and $D\phi_t$ is uniformly contracting on E^s , uniformly expanding on E^u , and the contraction (expansion) on E^c is strictly smaller than the one on E^s (E^u).

The invariant bundles $E^s, E^u, E^c, E^s \oplus E^c$ and $E^u \oplus E^c$ integrate to unique foliations W^s, W^u, W^c, W^{cs} and W^{cu} .

Theorem 2.2. (Spectral Decomposition for Flows) *If ϕ is an Anosov flow on the compact manifold M , then there exists a decomposition of the non-wandering set of ϕ , $NW(\phi) = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_k$, where the sets Λ_i are mutually disjoint, and each Λ_i is a transitive compact invariant set for ϕ , such that $\Lambda_i = \overline{W^{cs}(x)} \cap \overline{W^{cu}(x)}$, $\forall x \in \Lambda_i$.*

Throughout the paper, we fix the Anosov flow ϕ which satisfies the above theorem. We turn our attention now to perturbations of the time-one map of an Anosov flow. A partially hyperbolic diffeomorphism is dynamically coherent if the center-stable, central and center-unstable bundles integrate to invariant foliations W^{cs} , W^c and W^{cu} respectively, the center-stable foliation is sub-foliated by the center and the stable foliations, and the center-unstable foliation is sub-foliated by center and unstable foliations.

Since the center foliation is smooth for ϕ_1 , which is plaque expansive, hence, by Theorem 7.1 of Hirsch, Pugh and Shub (see [11]), there exists a C^1 neighborhood of ϕ_1 , such that every diffeomorphism in this neighborhood is partially hyperbolic with one dimensional center, and is dynamically coherent.

Proposition 2.3. *[11]/Theorem 7.1*

There exists a neighborhood \mathcal{U} of ϕ_1 in the space of C^1 diffeomorphisms, such that for each $f \in \mathcal{U}$, there is a homeomorphism $h_f : M \rightarrow M$, which is a leaf conjugacy, meaning that $h_f(W^c(x, \phi_1)) = W_f^c(h_f(x), f)$, and with h_f close to Id in the C^0 topology, where W_f^ denotes the $*$ foliation of f ($*$ = s, u, c, cs, cu).*

It is easy to see that the product structure of each basic set is preserved by the leaf conjugacy: the center stable leaves and center unstable leaves are mapped to the corresponding center stable leaves and center unstable leaves.

Corollary 2.4. *For any $f \in \mathcal{U}$ and any $1 \leq i \leq k$, $h_f(\Lambda_i)$ has local product structure, meaning that there exists $\beta_0 > 0$ such that if $x, y \in h_f(\Lambda_i)$, $d(x, y) < \beta <$*

β_0 , then $W_{2\beta}^u(x, f) \cap W_{2\beta}^{cs}(y, f)$ and $W_{2\beta}^s(x, f) \cap W_{2\beta}^{cu}(y, f)$ consist both of exactly one point which belongs to $h_f(\Lambda_i)$.

By the leaf conjugation, $f \in \mathcal{U}$ preserves each center leaf. For the flow ϕ , denote by $I(x)$ the flow segment $(\phi_t(x))_{t \in [0,1]}$. Then by the continuity, for each $f \in \mathcal{U}$, $f(x)$ is close to $\phi_1(x)$, and the segment $I_f(x) \subset W^c(x, f)$ between x and $f(x)$ is close to $I(x)$ in the Hausdorff topology. Moreover, $W^c(x, f) = \cup_i f^i(I_f(x))$ and the length of $I_f(x)$ is bounded uniformly from above and below by some positive numbers K_1 and K_2 , which are independent of x .

Let d_* be the distance between two points measured along W^* , and let $W_r^*(x, f)$ denote the ball of radius r centered at x in the $*$ -manifold of x with respect to f , where $*$ $\in \{s, c, u\}$.

As a consequence of the above discussion, one can easily show that

Lemma 2.5. *f is not expanding on the center leaves, meaning that there exist $K_1, K_2 > 0$ such that if $y \in W^c(x, f)$, with $d_c(x, y) \leq lK_1$ for some $l > 0$, then*

$$d_c(f^n(x), f^n(y)) \leq lK_2, \quad \forall n \geq 0.$$

The same is true for f^{-1} .

The following are various properties of the perturbations of the time-one map of an Anosov flow.

Proposition 2.6. *The non-wandering set of $f \in \mathcal{U}$ is contained in $\cup_{i=1}^k h_f(\Lambda_i)$. Each set $h_f(\Lambda_i)$ is saturated by center leaves and invariant under f .*

Proof. We only need to check that the non-wandering set of f is contained in $\cup_{i=1}^k h_f(\Lambda_i)$, the rest is an easy corollary of Proposition 2.3.

Fix $x \notin \cup_{i=1}^k \Lambda_i$, it suffices to show that $h_f(x)$ does not belong to the non-wandering set of f . Because x is wandering, there is a neighborhood U such that for each $y \in U$, its flow orbit intersects U with a unique connected component. This property is preserved clearly by the leaf conjugacy, that is, for every $h_f(y) \in h_f(U)$, $W_f^c(h_f(y))$ intersects $h_f(U)$ with a unique connected component with a small length. Then by Lemma 2.5, $f(h_f(y)) \notin h_f(U)$. This implies that $h_f(x)$ is a wandering point of f . □

Proposition 2.7. *The center holonomy between stable and unstable leaves along center-stable and respectively center-unstable leaves is globally defined, for every $f \in \mathcal{U}$: if $y \in W^c(x, f)$, then there exist the continuous holonomies $h_s^c : W^s(x, f) \rightarrow W^s(y, f)$ and $h_u^c : W^u(x, f) \rightarrow W^u(y, f)$ (we will use the notation h^c if no confusion can be made).*

Proof. First observe that restricted to any center unstable leaf, locally the center holonomy map is uniformly continuous:

Lemma 2.8. *For any $K > 0$, there exist ϵ_K such that if $f \in \mathcal{U}$, $y \in W_K^c(x, f)$, then the center holonomy map h^c is well defined from $W_{\epsilon_K}^*(x, f)$ to $W^*(y, f)$, $*$ $\in \{s, u\}$. Moreover, for any $0 < \epsilon < \epsilon_K$, there are $c_1(\epsilon, K), c_2(\epsilon, K) > 0$ such that:*

$$W_{c_1(\epsilon, K)}^*(y, f) \subset h^c(W_\epsilon^*(x, f)) \subset W_{c_2(\epsilon, K)}^*(y, f).$$

Furthermore $\lim_{\epsilon \rightarrow 0} c_1(\epsilon, K) = \lim_{\epsilon \rightarrow 0} c_2(\epsilon, K) = 0$.

Proof. The holonomy is clearly continuous, and by compactness it is uniformly continuous. \square

Now let us continue the proof.

Let $f \in \mathcal{U}$, and $x, y \in M$, with $y \in W^c(x, f)$. For any $N > 0$, we will construct the center holonomy map from $W_N^u(x, f)$ to $W^u(y, f)$.

We consider f^{-n} for n very large, then on one hand, the size of $f^{-n}(W_N^u(x, f))$ is very small, and on the other hand $d^c(f^{-n}(x), f^{-n}(y))$ is bounded uniformly from above by a number K , by Lemma 2.5. By (uniform) continuity of the center foliation of Lemma 2.8, we can construct the center-holonomy map h_n^c from $W_{\epsilon_K}^u(f^{-n}(x), f)$ to $W^u(f^{-n}(y), f)$, where ϵ_K is small enough (depending on K but not on n). Taking n large enough, we have that $f^{-n}(W_N^u(x, f)) \subset W_{\epsilon_K}^u(f^{-n}(x), f)$. Then push forward by f^n , and we obtain the holonomy map $h_u^c = f^n \circ h_n^c \circ f^{-n}$ from $W_N^u(x, f)$ to $W^u(y, f)$. Note that by the invariance of the center foliation, $h_{n-1}^c = f \circ h_n^c \circ f^{-1}$. We complete the proof by letting $N \rightarrow \infty$. The case of the center holonomy between stable leaves is similar.

We remark that the center holonomy map $h^c : W^u(x, f) \rightarrow W^u(y, f)$ satisfies that the distance $d_c(z, h^c(z))$ is uniformly bounded from above by a constant depending only on $d_c(x, y)$. This is because for n large enough,

$$d_c(f^{-n}(z), h_n^c(f^{-n}(z))) \approx d_c(f^{-n}(x), f^{-n}(y))$$

is bounded uniformly from above, then by Lemma 2.5, $d_c(z, h^c(z))$ is bounded uniformly from above. \square

The following proposition shows that in every center unstable leaf, the center foliation and the unstable foliation hold some kind of global product structure.

Proposition 2.9. $W^{cu}(x, f) = \cup_{y \in W^u(x, f)} W^c(y, f) = \cup_{y \in W^c(x, f)} W^u(y, f)$ for any $x \in M$ and $f \in \mathcal{U}$.

Proof. We need a property on the unstable holonomy in each center unstable leaf, which is similar to Proposition 2.7, hence we only give a sketch of its proof.

Lemma 2.10. *If $y \in W^u(x, f)$, then there exist the continuous holonomies $h_c^u : W^c(x, f) \rightarrow W^c(y, f)$.*

Proof. For $n \rightarrow \infty$, $f^{-n}(y) \rightarrow f^{-n}(x)$. Hence, there is $K_n \rightarrow \infty$, such that h_n^u is the unstable holonomy map from $W_{K_n}^c(f^{-n}(x), f)$ to $W^c(f^{-n}(y), f)$. Define $h_c^u = f^n \circ h_n^u \circ f^{-n}$ the unstable holonomy map from $f^n(W_{K_n}^c(f^{-n}(x), f))$ to $W^c(y, f)$. By Lemma 2.5, there exist $K_n' \rightarrow \infty$ such that $W_{K_n'}^c(x, f) \subset f^n(W_{K_n}^c(f^{-n}(x), f))$. Hence, h_c^u is well defined. \square

Now let us continue the proof. We first prove that

$$W^{cu}(x, f) = \cup_{y \in W^u(x, f)} W^c(y, f).$$

It is clear that $\cup_{y \in W^u(x, f)} W^c(y, f) \subset W^{cu}(x, f)$. Suppose by contradiction that $\cup_{y \in W^u(x, f)} W^c(y, f) \subsetneq W^{cu}(x, f)$, take $z \in \partial \cup_{y \in W^u(x, f)} W^c(y, f)$, by the local product structure of W^c and W^u in the center unstable leaf, it is easy to conclude that $W^c(z) \subset \partial \cup_{y \in W^u(x, f)} W^c(y, f)$ and $z \notin \cup_{y \in W^u(x, f)} W^c(y, f)$. Take $z' \in \cup_{y \in W^u(x, f)} W^c(y, f)$ close to z , then $W^u(z', f) \cap W^c(z, f) \neq \emptyset$. Note that $W^c(z', f) \cap$

$W^u(x, f) \neq \emptyset$. Hence, by the global holonomy of Proposition 2.7, $W^c(z, f) \cap W^u(x, f) \neq \emptyset$, which contradicts the fact that $z \notin \cup_{y \in W^u(x, f)} W^c(y, f)$.

The proof of $W^{cu}(x, f) = \cup_{y \in W^c(x, f)} W^u(y, f)$ is similar as above, replacing Proposition 2.7 by Lemma 2.10.

The proof of Proposition 2.9 is complete. \square

The following proposition is a generalization of a result in Plante [22]: for every Anosov transitive flow, if the strong unstable manifold of a periodic point is not dense, then this Anosov flow is a suspension flow with constant roof function.

Proposition 2.11. *For any $f \in \mathcal{U}$ there exists $K_0 > 0$ such that for any $x \in h_f(\Lambda_i)$, we have that $W_{K_0}^c(W^u(x, f), f)$ is dense in $h_f(\Lambda_i)$.*

Proof. By Proposition 2.9, for any $x \in h_f(\Lambda_i)$,

$$\cup_{L \in \mathbb{N}} (\cup_{z \in W_L^c(W_L^u(x, f), f)} W_1^s(z, f)) = \cup_{z \in W^{cu}(x, f)} W_1^s(z, f).$$

Because Λ_i is a basic component of the Anosov flow ϕ , $W^{cs}(h_f^{-1}(x), \phi)$ is dense in Λ_i . Then by the center conjugation of Proposition 2.6, $W^{cu}(x, f)$ is dense in $h_f(\Lambda_i)$ and

$$\{\Gamma_{f, L}(x) = \cup_{z \in W_L^c(W_L^u(x, f), f)} W_1^s(z, f)\}_{L \in \mathbb{N}}$$

is an open covering of the $h_f(\Lambda_i)$. Because $h_f(\Lambda_i)$ is compact, we can take $L(x)$ such that $h_f(\Lambda_i) \subset \Gamma_{f, L(x)}(x)$.

By the continuity of invariant foliations, for any $f \in \mathcal{U}$, and for any $x \in h_f(\Lambda_i)$, there exists a small neighborhood B_x of x such that for any $z \in B_x$, $h_f(\Lambda_i) \subset \Gamma_{f, L(x)}(z)$. Take an open covering B_{x_1}, \dots, B_{x_k} of $h_f(\Lambda_i)$ and let $L = \max\{L(x_1), \dots, L(x_k)\}$. By the above construction, for any $x \in h_f(\Lambda_i)$, we always have $h_f(\Lambda_i) \subset \Gamma_{f, L}(x)$.

For any $x \in h_f(\Lambda_i)$, we have that $h_f(\Lambda_i) \subset \Gamma_{f, L}(f^{-n}(x))$, and by the invariance of $h_f(\Lambda_i)$ we get $h_f(\Lambda_i) \subset f^n(\Gamma_{f, L}(f^{-n}(x)))$. Because the center is not expanding, there exists a constant $K_0 > 0$ depending only on L and \mathcal{U} (not on n) such that

$$f^n(\Gamma_{f, L}(f^{-n}(x))) \subset \cup_{z \in W_{K_0}^c(W^u(x, f), f)} f^n(W_1^s(f^{-n}(z), f)).$$

By the uniform contraction of the strong stable foliation, $W_{K_0}^c(W^u(x, f), f)$ is dense in $h_f(\Lambda_i)$. The proof is complete. \square

We remark that it is easy to show that K_0 from above can be chosen such that the result of the Proposition 2.11 is true for any $1 \leq i \leq k$ and for any g in a small neighborhood of f .

2.2. Topological entropy. In this subsection we will discuss about the topological entropy.

Definition 2.12. Let M be a compact metric space, and $f : M \rightarrow M$ a homeomorphism. For $K \subset M$ a subset not necessary invariant, the *topological entropy of f on K* is defined as follows. Let $d_n(x, y) = \max_{i=0}^{n-1} d(f^i(x), f^i(y))$. Let $a(f, n, \delta, K)$ be the maximal cardinality of δ -separated set of K in the metric d_n , and let $b(f, n, \delta, K)$ be the minimal cardinality of a δ -spanning set of K in the same metric d_n . Then

$$a(f, \delta, K) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log a(f, n, \delta, K),$$

$$b(f, \delta, K) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log b(f, n, \delta, K),$$

and the topological entropy of f on K is

$$h(f, K) = \lim_{\delta \rightarrow 0} a(f, \delta, K) = \lim_{\delta \rightarrow 0} b(f, \delta, K).$$

The topological entropy of f is $h(f) = h(f, M)$.

Remark 2.13. With the same hypothesis as above, suppose $M = \cup_{i=1}^{\infty} M_i$ is a measurable partition of M , then

$$h(f) = \sup_i h(f, M_i).$$

In the case of partially hyperbolic diffeomorphisms with one-dimensional center, and with non-expanding center, the entropy can be calculated using only the unstable manifolds.

Definition 2.14. Let f be a dynamically coherent partially hyperbolic diffeomorphism on M , then for any $x \in M$, we define

$$A(x, \delta) = \{y \in M : \exists x_u \in W_{\delta}^u(x, f), x_c \in W_{\delta}^c(x, f), y = W_{loc}^c(x_u, f) \cap W_{loc}^u(x_c, f)\},$$

and

$$D(x, \delta) = \cup_{y \in A(x, \delta)} W_{\delta}^s(y, f).$$

Proposition 2.15. *Let f be a dynamically coherent partially hyperbolic diffeomorphism on the compact manifold M , with one dimensional center, and Λ an invariant set of f with local product structure. Suppose that there exist $K_1, K_2 > 0$ such that if $y \in W_{K_1}^c(x)$ then $f^n(y) \in W_{K_2}^c(f^n(x))$, $\forall n \in \mathbb{N}$. Then there is $\delta_0 > 0$ such that for any $\delta < \delta_0$, $h(f, D(x, \delta) \cap \Lambda) = h(f, W_{\delta}^u(x) \cap \Lambda)$.*

Proof. We first prove that $h(f, W_{\delta}^u(x, f) \cap \Lambda) = h(f, A(x, \delta) \cap \Lambda)$.

One inequality is obvious, since $W_{\delta}^u(x, f) \cap \Lambda \subset A(x, \delta) \cap \Lambda$.

By the uniform transversity between the two bundles E^c and E^u , for any $\delta < \delta_0$ sufficiently small, in the definition of $A(x, \delta)$ above, $y = W_{\delta}^c(x_u, f) \cap W_{\delta}^u(x_c, f)$. We can choose $\delta_0 < K_1$. Note that Lemma 2.8 is still true in this case, let $c(\epsilon) = c_2(\epsilon, K_1)$ be the constant given there.

Let $S(n, \epsilon)$ be an (n, ϵ) -spanning set of $W_{\delta}^u(x) \cap \Lambda$ of minimal cardinality. For any $y \in S(n, \epsilon)$, let $T(y, n, \epsilon)$ be a $(n, c(\epsilon))$ -spanning set of $W_{K_1}^c(y) \cap A(x, \delta)$ of minimal cardinality. We claim that the cardinality of $T(y, n, \epsilon)$ is bounded by $\frac{2nK_2}{c(\epsilon)}$.

Proof. This is because the center manifold is one dimensional, and by hypothesis, $f^n(W_{K_1}^c(y, f)) \subset W_{K_2}^c(f^n(y), f)$. For each $0 \leq i \leq n$, we can take $T(f^i(y), 0, \epsilon)$ a $(0, c(\epsilon))$ -spanning set of $W_{K_2}^c(f^i(y), f)$ with cardinality bounded by $\frac{2K_2}{c(\epsilon)}$. Then $\cup_{i=0}^n f^{-i}(T(f^i(y), 0, \epsilon))$ forms the $(n, c(\epsilon))$ -spanning set we need. \square

Let $T(n, \epsilon) = \cup_{y \in S(n, \epsilon)} T(y, n, \epsilon)$. Then $|T(n, \epsilon)| < \frac{2nK_2}{c(\epsilon)} |S(n, \epsilon)|$ and one can easily show that $T(n, \epsilon)$ is $(n, 2c(\epsilon))$ -spanning for $A(x, \delta) \cap \Lambda$ (because of the local product structure of Λ).

So $b(f, n, 2c(\epsilon), A(x, \delta) \cap \Lambda) < \frac{2nK_2}{c(\epsilon)} b(f, n, \epsilon, W_{\delta}^u(x) \cap \Lambda)$, and taking the logarithm, dividing by n , and taking the limit for $n \rightarrow \infty$, we get that

$$b(f, 2c(\epsilon), A(x, \delta) \cap \Lambda) \leq b(f, \epsilon, W_{\delta}^u(x) \cap \Lambda).$$

Now taking $\epsilon \rightarrow 0$ and using that fact that $\lim_{\epsilon \rightarrow 0} c(\epsilon) = 0$ we get that $h(f, A(x, \delta) \cap \Lambda) \leq h(f, W_\delta^u(x) \cap \Lambda)$.

The proof that $h(f, D(x, \delta) \cap \Lambda) = h(f, A(x, \delta) \cap \Lambda)$ is similar (in fact it is easier, one uses the uniform contraction of the stable foliation). \square

3. PROOF OF THEOREM A

In this section we will give the proof of Theorem A. We fix $f \in \mathcal{U}$. First we need to improve Lemma 2.8.

Lemma 3.1. *Given $K > 0$, there exist $\epsilon_K, \gamma_K > 0$ and a neighborhood $\mathcal{U}_f \subset \mathcal{U}$ of f such that if $g \in \mathcal{U}_f$ and $y \in \cup_{z \in W_K^c(x, g)} B(z, \gamma_K)$, then the center-stable holonomy h^{cs} is well-defined between $W_{\epsilon_K}^u(x, g)$ and $W^u(y, g)$, and for any $\epsilon < \epsilon_K$, there exist $c_1(\epsilon, K), c_2(\epsilon, K) > 0$ such that*

$$W_{c_1(\epsilon, K)}^u(y, g) \subset h^{cs}(W_\epsilon^u(x, g)) \subset W_{c_2(\epsilon, K)}^u(y, g).$$

Proof. By Lemma 2.8, the center holonomy is well defined for f and $W_K^c(x)$; one can extend this holonomy to a neighborhood of f and a neighborhood of $W_K^c(x)$ using the continuity of the center-stable holonomy (and the compactness of M). The bound $c_1(\epsilon, K)$ and $c_2(\epsilon, K)$ come directly from the compactness. \square

Now we start the proof of the main Theorem. The proof is based on the uniform rate of expansion of unstable disks around the invariant sets $h_f(\Lambda_i)$. Because the non-wandering set of f is inside $\cup_{i=1}^k h_f(\Lambda_i)$, by variation principle, $h(f) = h(f, \cup_{i=1}^k h_f(\Lambda_i))$. We can assume without loss of generality that the entropy of f is $h(f) = h(f|_{h_f(\Lambda_1)})$. Denote $h_f(\Lambda_1) = \Lambda_f$. The following is the main lemma which we will use.

Lemma 3.2. *Let f, Λ_f be as above, and let $\epsilon > 0$. There exist $\delta > 0$ and $N(\epsilon) > 0$ such that for any $x \in \Lambda_f$, $f^{N(\epsilon)}(W_\delta^u(x, f))$ contains at least $\exp^{N(\epsilon)(h(f)-3\epsilon)}$ disjoint sets of the form $W_{2\delta}^u(y_i, f)$, with $y_i \in \Lambda_f$.*

Let us provide the idea of the proof first. The entropy of f is realized on some $W_\delta^u(x_0, f)$ for some δ small and $x_0 \in \Lambda_f$. For any $x \in \Lambda_f$, there exists an n_1 (independent of x) such that $f^{n_1}(W_\delta^u(x, f))$ contains a disk which is the image of cs -holonomy of $W_\delta^u(x_0, f)$ (the holonomy has the distance at most K_0 in the center direction and very small in the stable direction). Then a (n, ρ) -separated set in $W_\delta^u(x_0, f)$ can be translated to a $(n + n_1, c_1(\rho, \hat{K}_0))$ -separated set in $W_\delta^u(x, f)$ for some $\hat{K}_0 > 0$. By iterating n_2 more times, we obtain an $(n + n_1 + n_2, 2\delta)$ -separated set. Taking ρ arbitrarily small and n arbitrarily large, and using the fact that the entropy of f is attained on $W_\delta^u(x_0, f)$, we will obtain the result.

Proof. Let us recall that K_0 is given in Proposition 2.11. By Lemma 2.5, there is \hat{K}_0 such that for any $x \in M$ and $n \in \mathbb{Z}$, $f^n(W_{K_0}^c(x, f)) \subset W_{\hat{K}_0}^c(f^n(x), f)$. We may choose $\hat{K}_0 > K_0$. Taking $K = \hat{K}_0$ in Lemma 3.1, there exist $\epsilon_0 = \epsilon_{\hat{K}_0}$ and $\gamma_0 = \gamma_{\hat{K}_0}$. We claim that there is $L > 0$ such that for any $x \in \Lambda_f$, $W_{K_0}^c(W_L^u(x, f), f)$ is γ_0 dense in Λ_f .

Proof. For each point $x \in \Lambda_f$, by Proposition 2.11, there is $L_x > 0$ such that $W_{K_0}^c(W_{L(x)}^u(x, f), f)$ is γ_0 dense in Λ_f . In fact, there is a open neighborhood B_x such that for any $y \in B_x$, $W_{K_0}^c(W_{L(x)}^u(x, f), f)$ is γ_0 dense in Λ_f . Then $\{B_x\}_{x \in \Lambda_f}$

is an open cover. Take a finite subcover $\{B_{x_i}\}$ and let $L = \max\{L(x_i)\}$. The proof is complete. \square

Take $\delta = \min\{\epsilon_0, \delta_0\}$ where δ_0 is given by Proposition 2.15. Because Λ_f is compact, it can be covered by finitely many sets $D(x_i, \delta)$ with $x_i \in \Lambda_f$. Take a finite partition of $\Lambda_f = \cup_{i=1}^l P_i$ such that each component of this partition is contained in one of the above open set $D(x_j, \delta)$. By Remark 2.13, there is $1 \leq i \leq l$ such that $h(f) = h(f, P_i)$. Suppose that P_i is contained in $D(x_0, \delta)$, then $h(f) = h(f, D(x_0, \delta))$ for some $x_0 \in \Lambda_f$. Then by Proposition 2.15, $h(f) = h(f, W_\delta^u(x_0, f))$.

There is n_1 such that for any $x \in \Lambda_f$, $f^{n_1}(W_\delta^u(x, f)) \supset W_{L+c_2(\delta, K_0)}^u(f^{n_1}(x), f)$. By the claim above, $W_{K_0}^c(f^{n_1}(W_\delta^u(x, f)))$ is γ_0 dense in Λ_f . By Lemma 3.1, there is a center stable holonomy map $h_{x_0}^{cs}$ between $W_\delta^u(x_0, f)$ and $W_{c_2(\delta, K_0)}^u(z, f) \subset f^{n_1}(W_\delta^u(x, f))$.

Choose $0 < \rho < \delta$ such that

$$a(f, \rho, W_\delta^u(x_0, f)) > h(f, W_\delta^u(x_0, f)) - \epsilon = h(f) - \epsilon.$$

Take n_2 such that for any $y \in \Lambda_f$, $f^{n_2}(W_{c_1(\rho/2, K_0)}^u(y, f)) \supset W_{2\delta}^u(f^{n_2}(y), f)$ where $c_1(\rho/2, K_0)$ is given by Lemma 3.1. There exists n_ϵ , large enough so

$$\frac{n_\epsilon}{n_1 + n_\epsilon + n_2} > \frac{h(f) - 3\epsilon}{h(f) - 2\epsilon},$$

and such that we have $a(f, n_\epsilon, \rho, W_\delta^u(x_0, f)) > \exp^{n_\epsilon(h(f)-2\epsilon)}$. Let $N(\epsilon, \delta) = n_1 + n_\epsilon + n_2$.

Let S be a maximal (n_ϵ, ρ) -separated set in $W_\delta^u(x_0, f)$. Then

$$|S| = a(f, n_\epsilon, \rho_0, W_{\delta_0}^u(x_0, f)) > \exp^{n_\epsilon(h(f)-2\epsilon)},$$

and by the choice of n_ϵ we get $|S| > \exp^{N(\epsilon, \delta)(h(f)-3\epsilon)}$.

Because W^u is expanding, this means that $\forall a, b \in S$, $d_u(f^{n_\epsilon}(a), f^{n_\epsilon}(b)) > \rho$. Then $W_{\rho/2}^u(f^{n_\epsilon}(a), f) \cap W_{\rho/2}^u(f^{n_\epsilon}(b), f) = \emptyset$. Denote by $a' = h_{x_0}^{cs}(a) \in f^{n_1}(W_\delta^u(x, f))$, then there is a piecewise smooth curve $l(a) = l^c(a) \cup l^s(a)$ connecting a and a' such that $l^c(a) \subset W_{K_0}^c(a', f)$ and $l^s(a) \subset W_{\gamma_0}^s(a, f)$. This implies that $f^{n_\epsilon}(a)$ and $f^{n_\epsilon}(a')$ is connected by $f^{n_\epsilon}(l(a)) = f^{n_\epsilon}(l^c(a)) \cup f^{n_\epsilon}(l^s(a))$, where $f^{n_\epsilon}(l^c(a)) \subset W_{K_0}^c(f^{n_\epsilon}(a))$ by Lemma 2.5, and $f^{n_\epsilon}(l^s(a)) \subset W_{\gamma_0}^s(f^{n_\epsilon}(a'))$ by the uniformly contraction along the strong stable bundle. Hence, by Lemma 3.1, there is a holonomy map $h_{f^{n_\epsilon}(a)}^{cs} = f^{n_\epsilon} \circ h_{x_0}^{cs} \circ f^{-n_\epsilon}$ from $W_{\rho/2}^u(f^{n_\epsilon}(a), f)$ to $W_{c_2(\rho/2, K_0)}^u(f^{n_\epsilon}(a'), f)$, such that

$$h_{f^{n_\epsilon}(a)}^{cs}(W_{\rho/2}^u(f^{n_\epsilon}(a), f)) \supset W_{c_1(\rho/2, K_0)}^u(f^{n_\epsilon}(a'), f).$$

Then, there is a family of distinct unstable disks $\{W_{c_1(\rho/2, K_0)}^u(f^{n_\epsilon}(a'), f)\}_{a \in S}$, which are contained in $f^{n_1+n_\epsilon}(W_\delta^u(x, f))$. By the choice of n_2 and definition of $N(\epsilon, \delta)$, $f^{N(\epsilon, \delta)}(W_\delta^u(x, f))$ contains $\exp^{N(\epsilon, \delta)(h(f)-3\epsilon)}$ disjoint sets

$$\{W_{2\delta}^u(f^{n_\epsilon+n_2}(a'), f)\}_{a' \in h_{x_0}^{cs}(S)}.$$

And by the product structure of Λ_f , $f^{n_\epsilon+n_2}(a') \in \Lambda_f$. The proof is complete. \square

Using the continuity of the foliations we get the following corollary.

Corollary 3.3. *Let f, Λ_f be as above, and let $\epsilon, \delta > 0$, and $N(\epsilon, \delta)$ given by the previous lemma. There exists a neighborhood $\mathcal{U}_f(\epsilon) \subset \mathcal{U}$ of f such that for any $g \in \mathcal{U}_f(\epsilon)$, and for any $x \in \Lambda_g$ (corresponding to Λ_f), then $g^{N(\epsilon, \delta)}(W_\delta^u(x, g))$ contains at least $\exp^{N(\epsilon, \delta)(h(f) - 3\epsilon)}$ disjoint sets of the form $W_\delta^u(y_i, g)$, with $y_i \in \Lambda_g$.*

Now we can finish the proof of the main theorem. By induction we can show that for any $g \in \mathcal{U}_f(\epsilon)$, any $x \in \Lambda_g$, and any $l > 0$, then $g^{lN(\epsilon, \delta)}(W_\delta^u(x, g))$ contains at least $\exp^{lN(\epsilon, \delta)(h(f) - 3\epsilon)}$ disjoint sets of the form $W_\delta^u(y_i, g)$ for $y_i \in \Lambda_g$. But this gives rise to a $(lN(\epsilon, \delta), \delta)$ -separated set of $W_\delta^u(x, g)$ of cardinality greater or equal to $\exp^{lN(\epsilon, \delta)(h(f) - 3\epsilon)}$, so $h(g) \geq a(g, \delta, W_\delta^u(x, g)) \geq h(f) - 3\epsilon$, which shows the lower semicontinuity of the entropy function h .

The upper semicontinuity for partially hyperbolic diffeomorphisms with one-dimensional center bundle is proved in [17]. We complete the proof of the Theorem A.

4. APPENDIX: OTHER PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

Here we show that the topological entropy depends continuously in the case of the other known partially hyperbolic diffeomorphisms with one dimensional center mentioned in Introduction.

- (1) On the uniformly hyperbolic diffeomorphisms case, this diffeomorphism is structure stable, i.e., it is conjugate to nearby diffeomorphisms. Hence, the topological entropy is locally constant.
- (2) If f is a skew products over (topologically) uniformly hyperbolic base, as an immediate corollary of Ledrappier-Walters variational principle [16], the topological entropy of f and nearby diffeomorphisms coincide to the topological entropy on the base, which is topologically uniformly hyperbolic homeomorphisms, and the topological entropy is locally constant. For a more complete discussion see [25].

Let us note that, by the work of Parwani, Hammerlindl and Potrie ([21, 9, 10]), this case contains all partially hyperbolic diffeomorphisms on 3 dimension Nilmanifold where the manifold is different to T^3 .

- (3) A 3 dimensional partially hyperbolic diffeomorphism over T^3 is called derived Anosov if it is in the same isotopy class with a linear Anosov diffeomorphism. The first derived Anosov example was given by Mañé [18] to construct the example of diffeomorphisms which are transitive but not hyperbolic. In the Mañé's example, the diffeomorphisms are C^0 close to the linear Anosov diffeomorphism. Recent study on general derived Anosov diffeomorphisms can be found in [3, 8, 23, 10, 7].

The topological entropy of a 3 dimensional derived from Anosov diffeomorphism has the same topological entropy as the linear Anosov diffeomorphism. For the original Mañé's example [18], this fact was proved by [6]. For general derived Anosov diffeomorphisms, this was observed by Ures in [27] (for a generalization to higher dimension see [7]).

- (4) The non-dynamically coherent 3 dimensional examples of Hertz-Hertz-Ures [26] is in fact an Axiom A diffeomorphism with a torus attractor and a torus repeller. By the Ω stability theorem, its non-wandering set is conjugate to the non-wandering set of nearby diffeomorphisms. This implies that the topological entropy is locally constant.

- (6) Let us observe that the new 3 dimensional examples by Bonatti-Parwani-Potrie are perturbations of a partially hyperbolic diffeomorphism f_0 where f_0 has a unique attractor A and a unique repeller R , and $f|_{A \cup R}$ is the time-one map of a hyperbolic flow X_t restricted to basic pieces ([1][Theorem 1.1]). It is shown that f_0 is dynamically coherent and the center foliation is plaque expansive (definition see [11]). Hence, by [11], there is a C^1 neighborhood \mathcal{U} of f_0 such that every diffeomorphism $f \in \mathcal{U}$ is dynamically coherent. Moreover, the center foliation of f is structural stable, there is a homeomorphism h_f such that for any $x \in M$, $h_f(W^c(x, f_0)) = W^c(h_f(x), f)$ and $d_{C^0}(h, Id)$ is close to zero.

Denote by $A_f = h_f(A)$ and $R_f = h_f(R)$, both A_f and R_f are compact invariant and center saturated subset of f . Because A (resp. R) is an attractor (resp. repeller) for hyperbolic flow X_t , it is cu (resp. cs) saturated, which means, contain entire center unstable (resp. center stable) leaves of the corresponding diffeomorphism. Then by the leaf conjugacy, A_f and R_f are cu and cs saturated respectively.

Lemma 4.1. *For any $f \in \mathcal{U}$, the non-wandering set of f is contained in $A_f \cup R_f$.*

Proof. Here we only sketch the proof. Take a small neighborhood U of A , by continuation, for \mathcal{U} sufficiently small, $A_f \subset U$. It suffices to show that for every $x \in U \setminus A_f$, there is $n(x)$ such that $f^{-n(x)}(x) \notin U$.

Note that A_f is W_f^{cu} saturated, take $\epsilon > 0$ such that $U \subset \cup_{y \in A_f} W_\epsilon^s(y, f)$. For $z \in U$, define function

$$k(z) = \min_{y \in A_f \cap W^s(z, f)} \{d_s(y, z)\}.$$

Then for any $z \in U \setminus A_f$, $0 < k(z) < \epsilon$. There is $0 < \lambda < 1$ such that $\|Df|_{E^s(y)}\| < \lambda$ for any $y \in U$, then if $f^{-1}(x) \in U$, we have $k(f^{-1}(x)) > \frac{\epsilon}{\lambda}$. And by induction, if suppose $f^{-i}(x) \in U$ for every $i \geq 0$, $k^s(f^{-i}(x)) \rightarrow \infty$, which is a contradiction.

The proof is complete. □

Because A_f is cu saturated, it is easy to check that the previous arguments for flow case still work, we conclude that $h(f, A_f)$ varies continuously respect to $f \in \mathcal{U}$. For R_f , we will restate the proof by using its center stable leaves.

Let us remark in the end that the method used in this paper for the proof of the continuity of the entropy can be generalized to other partially hyperbolic diffeomorphisms with the dimension of the center equal to one. The fact that we considered perturbations of time-one maps of Anosov flows made the arguments easier, but in fact all that is needed is that the expansion of different unstable disks of points from the non-wandering set are comparable, and this should be true under the assumption of non-expanding center, dynamical coherence, and some kind of ‘weak’ transitivity for example.

REFERENCES

- [1] C. Bonatti, K. Parwani and R. Potrie. *Anomalous partially hyperbolic diffeomorphisms I: dynamically coherent examples*, Preprint arXiv:1411.1221.

- [2] C. Bonatti and A. Wilkinson. *Transitive partially hyperbolic diffeomorphisms on 3-manifolds*, *Topology* 44, no. 3, 2005, pp. 475-508.
- [3] M. Brin, D. Burago and S. Ivanov, *Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus*. *J. Mod. Dyn* 3, 2009, pp. 1-11.
- [4] M. Brin and Ya. Pesin. *Partially hyperbolic dynamical systems*. *Izv. Acad. Nauk. SSSR* 1, 1974, pp. 177-212.
- [5] J. Buzzi. *Intrinsic ergodicity for smooth interval maps*, *Israel J. Math* 100, 1997, pp. 125-161.
- [6] J. Buzzi, T. Fisher, M. Sambarino and C. Vásquez. *Maximal entropy measures for certain partially hyperbolic, derived from Anosov systems*. *Ergodic Theory and Dynamical Systems*, 32, 2011 pp 63-79.
- [7] T. Fisher, R. Potrie and M. Sambarino. *Dynamical coherence for partially hyperbolic diffeomorphisms of tori isotopic to Anosov* *Mathematische Zeitschrift*, 278, 2014, pp 149-168.
- [8] A. Hammerlindl, *Leaf conjugacies in the torus*, *Ergodic Th. and Dynam. Sys.* 33, No.3, 2013, pp. 896-933.
- [9] A. Hammerlindl, *Partial hyperbolicity on 3-dimensional nilmanifolds*, *Discrete and Continuous Dynamical Systems A*, 33:8, 2013, pp. 3641-3669.
- [10] A. Hammerlindl and R. Potrie. *Pointwise partial hyperbolicity in 3-dimensional nilmanifolds*, *Journal of the London Math. Soc.* 89 (3), 2014, pp. 853-875.
- [11] M. Hirsch, C. Pugh and M. Shub. *Invariant foliations*. *Lecture Notes in Mathematics*, Vol. 583. Springer-Verlag, Berlin-New York, 1977.
- [12] H. Hu and J. Zhu *Quasi-stability of partially hyperbolic diffeomorphisms*, *Transactions of the A.M.S.* 366, 2014, pp. 3787-3804.
- [13] Y. Hua *Continuity of topological entropy at time-one maps of transitive Anosov flows*. PhD Thesis, Northwestern University, 2009. Proquest LLC.
- [14] Y. Hua, R. Saghin and Z. Xia. *Topological entropy and partially hyperbolic diffeomorphisms*, *Erg. Th. Dyn. Syst.* 28, no. 3, 2008, pp. 843-862.
- [15] A. Katok. *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, *Inst. Hautes Études Sci. Publ. Math.* No. 51, 1980, 137-173.
- [16] F. Ledrappier and P. Walters. *A relativised variational principle for continuous transformations*. in *Journal London Math. Soc.* (2) 16, 1977, no.3, pp. 568-576.
- [17] G. Liao, M. Viana, and J. Yang. *The entropy conjecture for diffeomorphisms away from tangencies*, *Journal of the E.M.S.*, Volume 15, Issue 6, 2013, pp. 2043-2060.
- [18] R. Mañé, *Contributions to the stability conjecture*, *Topology*, 17, 1978, pp. 383-396.
- [19] M. Misiurewicz. *On non-continuity of topological entropy*. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 19, 1971, pp 319-320.
- [20] M. Misiurewicz. *Diffeomorphism without any measure of maximal entropy*. *Bull. Acad. Pol. Sci.*, 21, 1973, pp. 903-910, Series sci. math, astr. et phys.
- [21] K. Parwani, *On 3-manifolds that support partially hyperbolic diffeomorphisms*, *Nonlinearity*, 23, 2010, pp 589-606.
- [22] J.F. Plante, *Anosov flows*, *Amer. J. Math.* 94, 1972, pp 729-755.
- [23] R. Potrie, *Partial hyperbolicity and foliations in T^3* , *Preprint (2012) arXiv:1206.2860*. To appear in *Journal of Modern Dynamics*.
- [24] C. Pugh and M. Shub. *Ergodicity of Anosov actions*. *Invent. Math.* 15, 1972, pp 1-23.
- [25] F. Rodriguez-Hertz, J. Rodriguez-Hertz, A. Tahzibi and R. Ures. *Maximizing measures for partially hyperbolic systems with compact center leaves*, *Erg. Th. Dyn. Syst.* 32, 2012, pp 825-839.
- [26] F. Rodriguez-Hertz, J. Rodriguez-Hertz, R. Ures. *A non-dynamically coherent example on T^3* , to appear in *Annales de l'Institut Henri Poincaré*.
- [27] R. Ures. *Intrinsic ergodicity of partially hyperbolic diffeomorphisms with hyperbolic linear part*, *Proceedings of the AMS*, 140, 2012, pp. 1973-1985.
- [28] Yano, K. *A remark on the topological entropy of homeomorphisms*. *Invent. Math.* 59, 1980, pp 215-220.
- [29] Y. Yomdin. *Volume growth and entropy*, *Israel J. Math.*, 57, 1987, pp. 285-300.

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